

# Separability of massive field equations for spin-0 and spin-1/2 charged particles in the general non-extremal rotating charged black holes in minimal five-dimensional gauged supergravity

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We continue to investigate the separability of massive field equations for spin-0 and spin-1/2 charged particles in the general, non-extremal, rotating, charged, Chong-Cvetič-Lü-Pope black holes with two independent angular momenta and a non-zero cosmological constant in minimal  $D = 5$  gauged supergravity theory. We show that the complex Klein-Gordon equation and the modified Dirac equation with the inclusion of an extra counter-term can be separated by variables into purely radial and purely angular parts in this general Einstein-Maxwell-Chern-Simons background spacetime. A second-order symmetry operator that commutes with the complex Laplacian operator is constructed from the separated solutions and expressed compactly in terms of a rank-2 Stäckel-Killing tensor which admits a simple diagonal form in the chosen pentad one-forms so that it can be understood as the square of a rank-3 totally anti-symmetric tensor. A first-order symmetry operator that commutes with the modified Dirac operator is expressed in terms of a rank-3 generalized Killing-Yano tensor and its covariant derivative. The Hodge dual of this generalized Killing-Yano tensor is a generalized principal conformal Killing-Yano tensor of rank-2, which can generate a “tower” of generalized (conformal) Killing-Yano and Stäckel-Killing tensors that are responsible for the whole hidden symmetries of this general, rotating, charged, Kerr-AdS black hole geometry. In addition, the first laws of black hole thermodynamics have been generalized to the case that the cosmological constant can be viewed as a thermodynamical variable.

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## I. INTRODUCTION

In 1976, Chandrasekhar [1] showed that the massive Dirac's equation is separable in the Kerr geometry using the Newman-Penrose's null-tetrad formalism. Subsequently, this work was further extended by Page [2] and Lee [3] to the case of a Kerr-Newman black hole. Later on, Carter and McLenaghan [4] found that the separability of the Dirac equation in the Kerr geometry, is related to the fact that the skew-symmetric tensor corresponding to the two-index Killing spinor admitted by the Kerr metric is a rank-2 anti-symmetric Killing-Yano tensor, and its square is just a second-order symmetric Stäckel-Killing tensor discovered by Carter [5]. What is more, using the Killing-Yano tensor, they had constructed a first-order differential operator that commutes with the Dirac operator. The separation constant appearing in the separated solutions to the Dirac equation acts as the eigenvalue of this first order symmetry operator. The essential property that allows the construction of such a symmetry operator is the existence of a Killing-Yano tensor in the Kerr spacetime.

Recently, we [6, 7] have investigated the separability of Dirac's equation and its relation to the Killing-Yano tensor of rank-3 in the five-dimensional Myers-Perry [8] and Kerr-(anti-)de Sitter [9] black hole spacetime with two unequal angular momenta. A first-order symmetry operator commuting with the Dirac operator has been constructed by using the rank-3 Killing-Yano tensor whose square is just the rank-2 symmetric Stäckel-Killing tensor. In addition, we have constructed a second-order symmetry operator that commutes with the scalar Laplacian operator.

In a subsequent paper [10], we have extended this work to investigate the separability of a massive fermion field equation for spin-1/2 spinor particles in a five-dimensional rotating, charged, Cvetič-Youm [11] black hole with two different angular momenta and three equal charges. This black hole solution represents a natural generalization of the four-dimensional Kerr-Newman solution to five dimensions with the inclusion of a Chern-Simons term to the Maxwell equation, and belongs to the classes of Einstein-Maxwell-Chern-Simons (EMCS) black holes within minimal  $D = 5$  ungauged supergravity theory. It has been shown that the usual massive Dirac equation can not be separated by variables into purely radial and purely angular parts in this general rotating, charged black hole spacetime with two independent angular momenta. However, if an additional counter-term is supplemented into the usual Dirac operator, then the modified Dirac field equation for the spin-1/2 spinor particles is separable in this five-dimensional rotating charged black hole background geometry. A first-order symmetry operator that commutes with the modified

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Dirac operator has exactly the same form as that previously found in the uncharged Myers-Perry black hole case. This operator is expressed in terms of a rank-3 totally anti-symmetric tensor and its covariant derivative. The anti-symmetric tensor obeys a generalized Killing-Yano equation and its square is a second-order symmetric Stäckel-Killing tensor admitted by the five-dimensional rotating charged black hole spacetime. Furthermore, the inclusion of such an additional counter-term can be geometrically understood [12] as a natural consequence if one identifies the dual Maxwell three-form with a generalized “torsion”. This “formal” identification has many appealing features, for example, the Maxwell-Chern-Simons equations can be “simplified” to the standard Maxwell equations, and the generalized (conformal) Killing-Yano tensors “possess” many of the properties of the standard ones. However, it will unavoidably complicate the Einstein’s gravitational part since the inclusion of a torsion means that one has to deal with an asymmetric connection, therefore it is not necessary to make such a geometric identification.

In this paper, we will demonstrate that our previous analysis done in [10] is directly applicable to deal with the case of a nonzero cosmological constant, namely, the five-dimensional Chong-Cvetič-Lü-Pope [13] (alternatively, EMCS-Kerr-anti de Sitter) black hole. More specifically, we shall study the separation of variables for the massive field equations of spin-0 and spin-1/2 charged particles in this five-dimensional, general, non-extremal, rotating, charged EMCS-Kerr-AdS black holes with two independent angular momenta and a negative cosmological constant. This black hole metric is an exact solution within the framework of minimal  $D = 5$  gauged supergravity theory, and has attracted a lot of interest [14, 15, 16, 17, 18] since its being constructed. It has been shown [14, 15] that the geodesic equation, the Hamilton-Jacobi equation, the Klein-Gordon equation, and the stationary string in this spacetime are all separable since Frolov and his collaborators [19] initiated the studies (see [20] for a review and references therein) of hidden symmetries and separability properties of higher-dimensional rotating black hole spacetimes. The separability properties of these equations are also intimately associated with the existence of a second-order symmetric Stäckel-Killing tensor. In particular, the authors of Ref. [14] have demonstrated that the usual Dirac equation allows the separation of variables only in the special case with two equal magnitude angular momenta.

Therefore, the question of the separability of a spin-1/2 field equation and its relation to the rank-3 antisymmetric Killing-Yano tensor in the general Chong-Cvetič-Lü-Pope black hole background spacetime remains unsolved in a satisfactory manner similar to the uncharged case before we briefly announced in the conclusion section of [10] that the whole hidden symmetries of this general EMCS-Kerr-AdS black hole geometry are controlled by a rank-3 generalized Killing-Yano tensor and its Hodge dual two-form — a generalized principal conformal Killing-Yano tensor of rank-2. To resolve this question constitutes one of the main subjects of this paper. Specifically speaking, we will study the separation of variables for massive field equations of spin-0 and spin-1/2 charged particles in the general, non-extremal, rotating, charged, Chong-Cvetič-Lü-Pope black holes with two independent angular momenta and a non-zero cosmological constant. What is more, we shall present two symmetry operators that commutes respectively with the complex scalar Laplacian operator and the modified Dirac operator. These differential operators are, respectively, expressed in terms of a rank-2 Stäckel-Killing tensor and its “square root” — a rank-3 generalized Killing-Yano tensor. They characterize the separation constants that appear in the separable solutions of the massive Klein-Gordon scalar field equation and the modified Dirac equation with the minimal gauge coupling term. The Hodge dual of this generalized Killing-Yano tensor is a generalized principal conformal Killing-Yano tensor of rank-2, which can generate the “tower” of generalized Killing-Yano and Stäckel-Killing tensors that are responsible for the whole hidden symmetries of the general EMCS-Kerr-AdS black hole geometry. The separability properties of these wave equations are shown to be closely connected with the existence of these Stäckel-Killing and generalized (conformal) Killing-Yano tensors admitted by the five-dimensional EMCS-Kerr-AdS metric.

Our paper is organized as follows. In Sec. II, a simple, elegant form for the line element of the five-dimensional Chong-Cvetič-Lü-Pope black hole solution is presented in the Boyer-Lindquist coordinates, which is very convenient for us to explicitly construct the local Lorentzian orthonormal coframe one-forms (pentad). We also present some interesting properties of this spacetime. In particular, we will generalize the first law of black hole thermodynamics to the case when the cosmological constant is viewed as a thermodynamical variable. Section III summarizes the main results reported in this paper. In this section, we will discuss the hidden symmetries of the five-dimensional Chong-Cvetič-Lü-Pope black hole and present the explicit expressions of a rank-2 Stäckel-Killing tensor, a rank-3 generalized Killing-Yano tensor, a rank-2 generalized conformal Killing-Yano tensor. A second-order symmetry operator that commutes with the complex scalar Laplacian operator and a first-order differential operator that commutes with the modified Dirac operator are, respectively, expressed in terms of the rank-2 Stäckel-Killing tensor and the rank-3 generalized Killing-Yano tensor. Then in Sec. IV, we focus on the separation of variables for a massive complex Klein-Gordon scalar field equation in this spacetime background and use the separated solutions to directly construct a second-order differential operator that commutes with the complex scalar Laplacian operator, from which we can extract a concise expression for the rank-2 Stäckel-Killing tensor. Section V is devoted to the separation of variables for a massive spinor field equation in the general five-dimensional EMCS-Kerr-AdS black hole geometry. Adopting the fünfbein formalism, we will show that the modified Dirac’s equation with a minimal gauge coupling term can be separated into purely radial and purely angular parts. In Sec. VI, we shall demonstrate that the dual first order

differential operator previously constructed in terms of the rank-3 generalized Killing-Yano tensor (and its covariant derivative) has an eigenvalue as the separation constant appearing in the separated parts of the modified Dirac's equation, which means that this first-order symmetry operator commutes with the modified Dirac operator. The last section VII ends up with a summary of our work done in this paper. In the appendix, we give various useful expressions for the details of our calculations. The spin-connection one-forms are calculated by the first Cartan's structure equation from the exterior differential of the pentad. The corresponding spinor-connection one-forms, the Riemann curvature two-forms, and the Weyl curvature two-forms are also given in this pentad formalism.

## II. CHONG-CVETIČ-LÜ-POPE BLACK HOLE SOLUTION AND ITS SOME FUNDAMENTAL PROPERTIES

Although the neutrally-charged generalizations of the Kerr metric to higher dimensions were obtained [8] many years ago, higher-dimensional charged generalizations of the four-dimensional Kerr-Newman black hole still remain unknown in pure Einstein-Maxwell theory. In the simplest  $D = 5$  case, the inclusion of a Chern-Simons term to the Maxwell equation makes it easier to solve the field equations in the minimal supergravity theory. Up to now, almost all exact solutions known for rotating charged black holes in five dimensions fall into the framework of EMCS supergravity theory.

The bosonic part of minimal  $D = 5$  gauged supergravity theory consists of the metric and a one-form gauge field, and is given by

$$S = \frac{1}{16\pi} \int d^5x \left[ \sqrt{-g} (R + 12/l^2 - F_{\mu\nu} F^{\mu\nu}) - \frac{2}{3\sqrt{3}} \epsilon^{\mu\nu\alpha\beta\gamma} F_{\mu\nu} F_{\alpha\beta} \mathcal{A}_\gamma \right]. \quad (1)$$

Variation of this action yields the Einstein equation and the Maxwell-Chern-Simons equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{6}{l^2} g_{\mu\nu} = 2T_{\mu\nu} \equiv 2 \left( F_{\mu\alpha} F_\nu{}^\alpha - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right), \quad (2)$$

$$\partial_\nu \left( \sqrt{-g} F^{\mu\nu} + \frac{1}{\sqrt{3}} \epsilon^{\mu\nu\alpha\beta\gamma} \mathcal{A}_\alpha F_{\beta\gamma} \right) = 0. \quad (3)$$

A general solution that describes a five-dimensional non-extremal rotating, charged black hole with two independent angular momenta and a negative cosmological constant was constructed in Ref. [13]. The metric and the gauge potential given below simultaneously solve the Einstein equation and the Maxwell-Chern-Simons equation. We present here an elegant expression for the line element that had already been obtained by the present author soon after the announcement of this solution in the e-print archive.

For our purpose in this paper, we rewrite the metric for the five-dimensional rotating charged EMCS-Kerr-AdS black hole solution into a quasi-diagonal form so that we can easily construct a local Lorentzian orthonormal pentad with which the spinor field equation for the spin-1/2 particles can be decoupled into purely radial and purely angular parts. As did in Ref. [10], we find that the line element can be recast into a simple form in terms of the Boyer-Lindquist coordinates as follows:

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= -\frac{\Delta_r}{\Sigma} X^2 + \frac{\Sigma}{\Delta_r} dr^2 + \frac{\Sigma}{\Delta_\theta} d\theta^2 + \frac{\Delta_\theta (a^2 - b^2)^2 \sin^2 \theta \cos^2 \theta}{p^2 \Sigma} Y^2 + \left( \frac{ab}{rp} Z + \frac{Qp}{r\Sigma} X \right)^2, \end{aligned} \quad (4)$$

and the gauge potential is

$$\mathcal{A} = \frac{\sqrt{3}Q}{2\Sigma} X, \quad (5)$$

where we denote

$$X = dt - \frac{a \sin^2 \theta}{\chi_a} d\phi - \frac{b \cos^2 \theta}{\chi_b} d\psi, \quad (6a)$$

$$Y = dt - \frac{(r^2 + a^2)a}{(a^2 - b^2)\chi_a} d\phi - \frac{(r^2 + b^2)b}{(b^2 - a^2)\chi_b} d\psi, \quad (6b)$$

$$Z = dt - \frac{(r^2 + a^2) \sin^2 \theta}{a\chi_a} d\phi - \frac{(r^2 + b^2) \cos^2 \theta}{b\chi_b} d\psi, \quad (6c)$$

and

$$\begin{aligned}\Delta_r &= (r^2 + a^2)(r^2 + b^2) \left( \frac{1}{r^2} + \frac{1}{l^2} \right) - 2M + \frac{Q^2 + 2Qab}{r^2}, & \Delta_\theta &= 1 - \frac{p^2}{l^2}, \\ \Sigma &= r^2 + p^2, & p &= \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}, & \chi_a &= 1 - \frac{a^2}{l^2}, & \chi_b &= 1 - \frac{b^2}{l^2}.\end{aligned}$$

Here the parameters  $(M, Q, a, b, l)$  are related to the mass, two independent angular momenta of the black hole, and the cosmological constant.

The new form of the five-dimensional EMCS-Kerr-AdS metric (4) admits a simple, diagonal form:

$$ds^2 = \eta_{AB} e^A \otimes e^B = -(e^0)^2 + (e^1)^2 + (e^2)^2 + (e^3)^2 + (e^5)^2, \quad (7)$$

which allows us to choose the following local Lorentzian basis one-forms (pentad) defined as  $e^A = e^A_\mu dx^\mu$  orthonormal with respect to the flat (Lorentzian) metric  $\eta_{AB} = \text{diag}(-1, 1, 1, 1, 1)$ ,

$$e^0 = \sqrt{\frac{\Delta_r}{\Sigma}} X, \quad e^1 = \sqrt{\frac{\Sigma}{\Delta_r}} dr, \quad e^2 = \sqrt{\frac{\Sigma}{\Delta_p}} dp, \quad e^3 = \sqrt{\frac{\Delta_p}{\Sigma}} Y, \quad e^5 = -\left( \frac{ab}{rp} Z + \frac{Qp}{r\Sigma} X \right). \quad (8)$$

Throughout this paper, we shall adopt conventions as follows: Greek letters  $\mu, \nu$  run over five-dimensional space-time coordinate indices  $\{t, r, \theta, \phi, \psi\}$ , while Latin letters  $A, B$  denote local orthonormal (Lorentzian) frame indices  $\{0, 1, 2, 3, 5\}$ .

The above line element (4) is written in a coordinate frame rotating at infinity. To compute the physical mass, angular momenta and electric charge, one has to change the metric to a coordinate frame nonrotating at infinity by making the transformations:  $\phi = \tilde{\phi} - at/l^2$  and  $\psi = \tilde{\psi} - bt/l^2$ .

The outer event horizon is determined by the largest root of  $\Delta_{r_+} = 0$ . The Hawking temperature  $T = \kappa/(2\pi)$  and the Bekenstein-Hawking entropy  $S = A/4$  with respect to this horizon can be easily computed as

$$T = \frac{r_+^4 [1 + (2r_+^2 + a^2 + b^2)/l^2] - (Q + ab)^2}{2\pi r_+ [(r_+^2 + a^2)(r_+^2 + b^2) + Qab]}, \quad (9)$$

$$S = \pi^2 \frac{(r_+^2 + a^2)(r_+^2 + b^2) + Qab}{2\chi_a \chi_b r_+}, \quad (10)$$

while the angular velocities and the electro-static potential, measured relative to a frame that is non-rotating at infinity, are given by

$$\Omega_a = \frac{a(r_+^2 + b^2)(1 + r_+^2/l^2) + Qb}{(r_+^2 + a^2)(r_+^2 + b^2) + Qab}, \quad (11)$$

$$\Omega_b = \frac{b(r_+^2 + a^2)(1 + r_+^2/l^2) + Qa}{(r_+^2 + a^2)(r_+^2 + b^2) + Qab}, \quad (12)$$

$$\Phi = \frac{\sqrt{3}Qr_+^2/2}{(r_+^2 + a^2)(r_+^2 + b^2) + Qab}. \quad (13)$$

The physical mass, two angular momenta, and the electric charge are given by [13, 17]

$$\mathcal{M} = \frac{\pi}{2\chi_a \chi_b} \left[ \left( M + \frac{Qab}{l^2} \right) \left( \frac{1}{\chi_a} + \frac{1}{\chi_b} \right) - \frac{1}{2} M \right], \quad (14)$$

$$J_a = \frac{\pi [2Ma + Qb(2 - \chi_a)]}{4\chi_a^2 \chi_b}, \quad (15)$$

$$J_b = \frac{\pi [2Mb + Qa(2 - \chi_b)]}{4\chi_a \chi_b^2}, \quad (16)$$

$$\mathcal{Q} = \frac{\sqrt{3}\pi Q}{2\chi_a \chi_b}, \quad (17)$$

which fulfill the closed forms for the first law of black hole thermodynamics

$$\frac{2}{3} \mathcal{M} = TS + \Omega_a J_a + \Omega_b J_b + \frac{2}{3} \Phi \mathcal{Q} - \frac{1}{3} \Theta l, \quad (18a)$$

$$d\mathcal{M} = TdS + \Omega_a dJ_a + \Omega_b dJ_b + \Phi d\mathcal{Q} - \Theta dl, \quad (18b)$$

where we have introduced the generalized force conjugate to the cosmological radius  $l$  as

$$\Theta = \frac{\pi}{2\chi_a\chi_b l} \left[ M + \left( M + \frac{Qab}{l^2} \right) \left( \frac{1}{\chi_a} + \frac{1}{\chi_b} - \frac{3}{1+r_+^2/l^2} \right) - \frac{3Q^2}{2l^2(1+r_+^2/l^2)} \right]. \quad (19)$$

In practice, it is very useful to adopt  $p$  rather than  $\theta$  as the appropriate angle coordinate. What is more, the radial part and the angular part can be presented in a symmetric manner. In what follows, we shall adopt  $p$  as the convenient angle coordinate throughout this article. In doing so, the five-dimensional Chong-Cvetič-Lü-Pope metric can be rewritten as

$$ds^2 = -\frac{\Delta_r}{\Sigma} X^2 + \frac{\Sigma}{\Delta_r} dr^2 + \frac{\Sigma}{\Delta_p} dp^2 + \frac{\Delta_p}{\Sigma} Y^2 + \left( \frac{ab}{rp} Z + \frac{Qp}{r\Sigma} X \right)^2, \quad (20)$$

where

$$\Delta_p = -(p^2 - a^2)(p^2 - b^2) \left( \frac{1}{p^2} - \frac{1}{l^2} \right), \quad (21)$$

and

$$X = dt - \frac{(p^2 - a^2)a}{(b^2 - a^2)\chi_a} d\phi - \frac{(p^2 - b^2)b}{(a^2 - b^2)\chi_b} d\psi, \quad (22a)$$

$$Y = dt + \frac{(r^2 + a^2)a}{(b^2 - a^2)\chi_a} d\phi + \frac{(r^2 + b^2)b}{(a^2 - b^2)\chi_b} d\psi, \quad (22b)$$

$$Z = dt - \frac{(r^2 + a^2)(p^2 - a^2)}{(b^2 - a^2)a\chi_a} d\phi - \frac{(r^2 + b^2)(p^2 - b^2)}{(a^2 - b^2)b\chi_b} d\psi. \quad (22c)$$

After doing the following coordinate transformations:

$$t = \tau + (a^2 + b^2)u + a^2 b^2 v, \quad \phi = a\chi_a(u + b^2 v), \quad \psi = b\chi_b(u + a^2 v), \quad (23)$$

we get

$$X = d\tau + p^2 du, \quad Y = d\tau - r^2 du, \quad Z = d\tau + (p^2 - r^2)du - r^2 p^2 dv, \quad (24)$$

and find that the line element (20) possesses the same form recently used in [21].

The spacetime metric (4) is of Petrov type 22, like the three-equal-charge Cvetič-Youm black hole solution and its super-symmetric counterpart — the BMPV black hole solution. It possesses a pair of real principal null vectors  $\{\mathbf{l}, \mathbf{n}\}$ , a pair of complex principal null vectors  $\{\mathbf{m}, \bar{\mathbf{m}}\}$ , and one real, spatial-like unit vector  $\mathbf{k}$ . They can be constructed to be of Kinnersley-type as follows:

$$\begin{aligned} \mathbf{l}^\mu \partial_\mu &= \frac{1}{r^2 \Delta_r} \left[ (r^2 + a^2)(r^2 + b^2) \partial_t + (r^2 + b^2)a\chi_a \partial_\phi + (r^2 + a^2)b\chi_b \partial_\psi \right. \\ &\quad \left. + Q(ab\partial_t + b\chi_a \partial_\phi + a\chi_b \partial_\psi) \right] + \partial_r, \end{aligned} \quad (25a)$$

$$\begin{aligned} \mathbf{n}^\mu \partial_\mu &= \frac{1}{2r^2 \Sigma} \left[ (r^2 + a^2)(r^2 + b^2) \partial_t + (r^2 + b^2)a\chi_a \partial_\phi + (r^2 + a^2)b\chi_b \partial_\psi \right. \\ &\quad \left. + Q(ab\partial_t + b\chi_a \partial_\phi + a\chi_b \partial_\psi) \right] - \frac{\Delta_r}{2\Sigma} \partial_r, \end{aligned} \quad (25b)$$

$$\mathbf{m}^\mu \partial_\mu = \frac{\sqrt{\Delta_p/2}}{r + ip} \left[ \partial_p + i \frac{(p^2 - a^2)(p^2 - b^2)}{p^2 \Delta_p} \left( \partial_t - \frac{a\chi_a}{p^2 - a^2} \partial_\phi - \frac{b\chi_b}{p^2 - b^2} \partial_\psi \right) \right], \quad (25c)$$

$$\bar{\mathbf{m}}^\mu \partial_\mu = \frac{\sqrt{\Delta_p/2}}{r - ip} \left[ \partial_p - i \frac{(p^2 - a^2)(p^2 - b^2)}{p^2 \Delta_p} \left( \partial_t - \frac{a\chi_a}{p^2 - a^2} \partial_\phi - \frac{b\chi_b}{p^2 - b^2} \partial_\psi \right) \right], \quad (25d)$$

$$\mathbf{k}^\mu \partial_\mu = \frac{1}{rp} (ab\partial_t + b\chi_a \partial_\phi + a\chi_b \partial_\psi). \quad (25e)$$

These vectors satisfy the following orthogonal relations

$$\mathbf{l}^\mu \mathbf{n}_\mu = -1, \quad \mathbf{m}^\mu \bar{\mathbf{m}}_\mu = 1, \quad \mathbf{k}^\mu \mathbf{k}_\mu = 1, \quad (26)$$

and all others are zero. In terms of these vectors, the line element for the EMCS-Kerr-AdS black hole (4) can be put into a seminull pentad formalism (221 formalism) as follows:

$$ds^2 = -\mathbf{l} \otimes \mathbf{n} - \mathbf{n} \otimes \mathbf{l} + \mathbf{m} \otimes \bar{\mathbf{m}} + \bar{\mathbf{m}} \otimes \mathbf{m} + \mathbf{k} \otimes \mathbf{k}. \quad (27)$$

### III. HIDDEN SYMMETRIES OF CHONG-CVETIČ-LÜ-POPE BLACK HOLE AND TOWER OF GENERALIZED (CONFORMAL) KILLING-YANO TENSORS

In this section, we summarize our results about the complete hidden symmetry properties of general five-dimensional Chong-Cvetič-Lü-Pope black holes. In particular, we propose how to generalize the concepts of Killing-Yano and conformal Killing-Yano tensors so that they can be subject to the five-dimensional Einstein-Maxwell-Chern-Simons theory. To proceed, we first give a brief review on the recent work about the construction of the Stäckel-Killing tensor from the (conformal) Killing-Yano tensor.

Carter [5] found that the geodesic Hamilton-Jacobi equation and Klein-Gordon scalar field equation are separable by variables in the four-dimensional Kerr metric, and there exists an additional quadratic integral of motion, called as the Carter's fourth constant. This constant is associated with a second-order symmetric Stäckel-Killing tensor  $K_{\mu\nu} = K_{\nu\mu}$ , which obeys the Killing equation

$$K_{\mu\nu;\rho} + K_{\nu\rho;\mu} + K_{\rho\mu;\nu} = 0. \quad (28)$$

Penrose and Floyd [22] further discovered that this Stäckel-Killing tensor can be written in the form  $K_{\mu\nu} = f_{\mu\rho} f_{\nu}{}^{\rho}$ , where the skew-symmetric tensor  $f_{\mu\nu} = -f_{\nu\mu}$  is the Killing-Yano tensor obeying the equation  $f_{\mu\nu;\rho} + f_{\rho\mu;\nu} = 0$ . Using this object, Carter and McLenaghan [4] constructed a first-order symmetry operator that commutes with the massive Dirac operator. In the case of a  $D = 4$  Kerr black hole, the Killing-Yano tensor  $f$  is of rank-2, its Hodge dual  $k = *f$  is a rank-2, antisymmetric, conformal Killing-Yano tensor obeying the equation

$$k_{\alpha\beta;\gamma} + k_{\alpha\gamma;\beta} = g_{\alpha\beta}\xi_{\gamma} + g_{\gamma\alpha}\xi_{\beta} - 2g_{\beta\gamma}\xi_{\alpha}, \quad (29)$$

where the Killing vector is defined by

$$\xi_{\alpha} = \frac{1}{D-1} k^{\mu}{}_{\alpha;\mu}. \quad (30)$$

The above equation is equivalent to the Penrose's equation [23]

$$\mathcal{P}_{\alpha\beta\gamma} \equiv k_{\alpha\beta;\gamma} + g_{\beta\gamma}\xi_{\alpha} - g_{\gamma\alpha}\xi_{\beta} = 0. \quad (31)$$

A conformal Killing-Yano tensor  $k$  is dual to the Killing-Yano tensor if and only if it is closed  $dk = 0$ . This fact implies that there at least locally exists a potential one-form  $\hat{b}$  so that  $k = d\hat{b}$ . Carter [24] first found this potential to generate the Killing-Yano tensor for the Kerr-Newman black hole.

Recently, these results have been extended to higher-dimensional rotating, uncharged black hole solutions. In the special case of  $D = 5$  dimensions, it was demonstrated [19] that the rank-2 Stäckel-Killing tensor can be constructed from its "square root", a rank-3, totally antisymmetric Killing-Yano tensor. According to Carter's recipe [24], Frolov *et al.* [19] started from a potential one-form to generate a rank-2 conformal Killing-Yano tensor, whose Hodge dual  $f = *k$  is the expected rank-3 Killing-Yano tensor. The conformal Killing-Yano tensor  $k = d\hat{b}$ , the Killing-Yano tensor  $f = *k$ , and the Killing vector  $h = *(k \wedge k) = 2rp e^5$  constitute a tower of Killing-Yano tensors, and they are responsible for the whole hidden symmetries of  $D = 5$  Myers-Perry and Kerr-AdS black holes.

Now we focus on the general case of rotating, charged black holes in five dimensions. It is clear that all of the five-dimensional Myers-Perry black hole, Kerr-AdS black hole, three-equal-charge Cvetič-Youm black hole, and Chong-Cvetič-Lü-Pope black hole (4) possess  $R \times U(1)^2$  isometry group generated by three Killing vectors ( $\partial_t$ ,  $\partial_{\phi}$ , and  $\partial_{\psi}$ ). Besides, the separability properties of the geodesic equation, the Hamilton-Jacobi equation, and the Klein-Gordon equation in these black hole backgrounds imply that they are closely related to the existence of a rank-2 symmetric Stäckel-Killing tensor admitted by all these spacetime metrics. In the local Lorentzian coframe given in Eq. (8), we [6, 7, 10] find that the symmetric tensor  $K_{\mu\nu} = K_{\nu\mu}$  in any of these spacetimes has a simple, diagonal form

$$K_{AB} = \text{diag}(-p^2, p^2, -r^2, -r^2, p^2 - r^2), \quad (32)$$

which is equivalent to those previously given in [14, 15], up to an additive constant.

On the other hand, it has been [6, 7] shown that the separability of the Dirac's equation in the Myers-Perry and Kerr-AdS spacetime backgrounds is intimately associated with the existence of a rank-3 antisymmetric Killing-Yano tensor admitted by these uncharged metrics. What is more, it has been revealed that the symmetric Stäckel-Killing tensor given by Eq. (32) can be written as the square of this rank-3 Killing-Yano tensor

$$K_{\mu\nu} = -\frac{1}{2} f_{\mu\alpha\beta} f_{\nu}{}^{\alpha\beta}, \quad (33)$$

where the rank-3 Killing-Yano tensor is given by

$$f = (-p e^0 \wedge e^1 + r e^2 \wedge e^3) \wedge e^5 = {}^*k, \quad (34)$$

and satisfy the following Killing-Yano equation

$$f_{\alpha\beta\mu;\nu} + f_{\alpha\beta\nu;\mu} = 0. \quad (35)$$

The Hodge dual of the three-form  $f$  is a two-form  $k = {}^*f$ , which is a conformal Killing-Yano tensor obeying Eq. (29). Adopting the following definitions:

$$k_{\mu\nu} = -({}^*f)_{\mu\nu} = -\frac{1}{6}\sqrt{-g}\epsilon_{\mu\nu\alpha\beta\gamma}f^{\alpha\beta\gamma}, \quad f_{\alpha\beta\gamma} = ({}^*k)_{\alpha\beta\gamma} = \frac{1}{2}\sqrt{-g}\epsilon_{\alpha\beta\gamma\mu\nu}k^{\mu\nu}, \quad (36)$$

and the convention  $\epsilon^{01235} = 1 = -\epsilon_{01235}$  for the totally anti-symmetric tensor density  $\epsilon_{ABCDE}$ , we find that this two-form is

$$k = r e^0 \wedge e^1 + p e^2 \wedge e^3 = d\hat{b}, \quad (37)$$

which can be generated from a potential one-form

$$2\hat{b} = (p^2 - r^2)dt + \frac{(r^2 + a^2)(p^2 - a^2)a}{(b^2 - a^2)\chi_a}d\phi + \frac{(r^2 + b^2)(p^2 - b^2)b}{(a^2 - b^2)\chi_b}d\psi. \quad (38)$$

Clearly, the two-form  $k = {}^*f$  is closed,  $dk = d^2\hat{b} = 0$ , which indicates that  $f^{\mu\nu\rho}_{;\rho} = 0$ .

If the conformal Killing-Yano tensor  $k_{\mu\nu} = -k_{\nu\mu}$  is closed  $dk = 0$ , namely

$$k_{\alpha\beta;\gamma} + k_{\beta\gamma;\alpha} + k_{\gamma\alpha;\beta} = 0, \quad (39)$$

then the Penrose potential possesses the following properties:

$$\mathcal{P}_{\alpha\beta\gamma} + \mathcal{P}_{\beta\alpha\gamma} = 0, \quad (40a)$$

$$\mathcal{P}_{\alpha\beta\gamma} + \mathcal{P}_{\beta\gamma\alpha} + \mathcal{P}_{\gamma\alpha\beta} = 0, \quad (40b)$$

$$\mathcal{P}_{\alpha\beta}{}^{\beta} = 0 = \mathcal{P}_{\beta\alpha}{}^{\beta}. \quad (40c)$$

We now demonstrate how the concepts of Killing-Yano and conformal Killing-Yano tensors can be generalized to the charged case. Generally speaking, it is very complicated to find the concrete expressions for the Killing objects via solving the equations that they should be satisfied for the spacetime under consideration. Conversely, once given the analytical expressions for the Killing objects from the beginning, it is relatively easy and simple to check the equations that they obey. Therefore, we shall follow the latter routine. In other words, we first assume that the expected Killing-Yano tensors have the same form as the one in the uncharged case since they should recover it, then we examine their corresponding properties and find out the new equation that they should fulfill.

Just like in the case of the three-equal-charge Cvetič-Youm black hole, the charged Hamilton-Jacobi equation (essentially, the lowest order WKB approximation of Klein-Gordon equation) and the complex Klein-Gordon equation for a scalar field with rest mass  $\mu_0$  and electric charge  $q$

$$g^{\mu\nu}(\partial_\mu \mathcal{S} + q\mathcal{A}_\mu)(\partial_\nu \mathcal{S} + q\mathcal{A}_\nu) + \mu_0^2 = 0, \quad (41)$$

$$(\square_c - \mu_0^2)\Phi = (\nabla_\mu + iq\mathcal{A}_\mu)[g^{\mu\nu}(\nabla_\nu + iq\mathcal{A}_\nu)\Phi] - \mu_0^2\Phi = 0, \quad (42)$$

are separable for variables in the Chong-Cvetič-Lü-Pope black hole background geometry. The separability of these equations indicates that the five-dimensional EMCS-Kerr-AdS metric admits a rank-2 Stäckel-Killing tensor exactly given by Eq. (32). The separation constant acts as the eigenvalue of the dual operator

$$\mathbb{K}_c = (\nabla_\mu + iq\mathcal{A}_\mu)[K^{\mu\nu}(\nabla_\nu + iq\mathcal{A}_\nu)], \quad (43)$$

which commutes with the complex scalar Laplacian operator  $\square_c$ .

On the other hand, the separability of a spin-1/2 field equation [10]

$$(\tilde{\mathbb{H}}_D + \mu_e)\Psi = \left[ \gamma^\mu(\nabla_\mu + iq\mathcal{A}_\mu) + \frac{i}{4\sqrt{3}}\gamma^\mu\gamma^\nu F_{\mu\nu} + \mu_e \right]\Psi = 0, \quad (44)$$

in the three-equal-charge Cvetič-Youm black hole and the Chong-Cvetič-Lü-Pope spacetime backgrounds is also closely associated with the existence of a rank-3 antisymmetric tensor admitted by these spacetimes. The separation constant introduced in the modified Dirac equation behaves as the eigenvalue of a first-order differential operator [10]

$$\mathbb{H}_f = -\frac{1}{2}\gamma^\mu\gamma^\nu f_{\mu\nu}{}^\rho(\nabla_\rho + iq\mathcal{A}_\rho) + \frac{1}{16}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma f_{\mu\nu\rho;\sigma}, \quad (45)$$

that commutes with the modified Dirac operator  $\tilde{\mathbb{H}}_D$ . Here the minimal electro-magnetic coupling interaction has been taken into consideration.

Now that in the local Lorentzian coframe pentad (8), the Stäckel-Killing tensor has a simple, diagonal form given by Eq. (32), it is reasonable to assume that the expected anti-symmetric tensor of rank-3 is still given by Eq. (34) which can naturally reduce to that in the uncharged case [6, 7]. What is more, this anti-symmetric tensor still can be understood as the “square root” of the rank-2 Stäckel-Killing tensor via Eq. (33). It is also easy to find that the Hodge dual of this rank-3 tensor is still given by Eq. (37) and can be generated from a potential one-form (38), similar to the uncharged Myers-Perry and Kerr-AdS black hole cases.

At this stage, we are in a position to check whether these anti-symmetric tensors still satisfy the usual (conformal) Killing-Yano equation. In our previous work [10], we have proposed to generalize the concepts of the (conformal) Killing-Yano tensors so that they can be subject to the five-dimensional minimal gauged supergravity theory. To this end, we propose that a generalized conformal Killing-Yano tensor should satisfy the following equation

$$\mathcal{P}_{\alpha\beta\gamma} = \frac{1}{\sqrt{3}}\tilde{F}_{\alpha\beta}{}^\lambda k_{\gamma\lambda} = \frac{1}{\sqrt{3}}f_{\alpha\beta}{}^\lambda F_{\gamma\lambda}. \quad (46)$$

Equivalently, it can be rewritten as

$$k_{\alpha\beta;\gamma} + k_{\alpha\gamma;\beta} = g_{\alpha\beta}\xi_\gamma + g_{\gamma\alpha}\xi_\beta - 2g_{\beta\gamma}\xi_\alpha + \frac{1}{\sqrt{3}}(\tilde{F}_{\alpha\beta}{}^\lambda k_{\gamma\lambda} + \tilde{F}_{\alpha\gamma}{}^\lambda k_{\beta\lambda}), \quad (47)$$

where  $4\xi^\alpha = k^{\mu\alpha}{}_{;\mu}$ , and the dual Maxwell three-form is defined by

$$\tilde{F}_{\alpha\beta\gamma} = (*F)_{\alpha\beta\gamma} = \frac{1}{2}\sqrt{-g}\epsilon_{\alpha\beta\gamma\mu\nu}F^{\mu\nu}. \quad (48)$$

The cyclic property of the Penrose potential leads to the following important identities

$$k \wedge F = 0 = f \wedge \tilde{F}. \quad (49)$$

Since this rank-3 antisymmetric tensor  $f = *k$  is the Hodge dual of the two-form  $k = d\hat{b}$ , we can take the Hodge dual of the generalized Penrose equation (46) and obtain

$$\begin{aligned} f_{\alpha\beta\mu;\nu} &= \frac{1}{2}\sqrt{-g}\epsilon_{\alpha\beta\mu\rho\sigma}k^{\rho\sigma}{}_{;\nu} \\ &= \frac{1}{2}\sqrt{-g}\epsilon_{\alpha\beta\mu\rho\sigma}(\mathcal{P}^{\rho\sigma}{}_\nu + \delta_\nu^\rho\xi^\sigma - \delta_\nu^\sigma\xi^\rho) \\ &= \mathcal{W}_{\alpha\beta\mu\nu} + \sqrt{-g}\epsilon_{\alpha\beta\mu\nu\sigma}\xi^\sigma, \end{aligned} \quad (50)$$

where we have denoted

$$\mathcal{W}_{\alpha\beta\mu\nu} \equiv \frac{1}{2}\sqrt{-g}\epsilon_{\alpha\beta\mu\rho\sigma}\mathcal{P}^{\rho\sigma}{}_\nu = \frac{1}{2\sqrt{3}}\sqrt{-g}\epsilon_{\alpha\beta\mu\rho\sigma}f^{\rho\sigma\lambda}F_{\nu\lambda}. \quad (51)$$

Symmetrization of Eq. (50) with respect to the last two indices  $(\mu, \nu)$  gives a generalized Killing-Yano equation

$$f_{\alpha\beta\mu;\nu} + f_{\alpha\beta\nu;\mu} = \mathcal{W}_{\alpha\beta\mu\nu} + \mathcal{W}_{\alpha\beta\nu\mu}, \quad (52)$$

which had already been proposed in our previous work [10].

Finally, after a lengthy computations, we can work out the commutator

$$\begin{aligned} [\mathbb{H}_f, \tilde{\mathbb{H}}_D] &= \left[ \frac{1}{8}\gamma^\alpha\gamma^\beta\gamma^\nu(\nabla_\nu f_{\alpha\beta}{}^\mu + \nabla^\mu f_{\alpha\beta\nu}) + \frac{i}{2\sqrt{3}}(\gamma^\nu\gamma^\beta - \gamma^\beta\gamma^\nu)F_\nu{}^\alpha f_{\alpha\beta}{}^\mu \right](\nabla_\mu + iq\mathcal{A}_\mu) \\ &\quad - \frac{1}{16}\gamma^\mu\gamma^\alpha\gamma^\beta\gamma^\rho\gamma^\sigma(\nabla_\mu\nabla_\sigma f_{\alpha\beta\rho} + 2f_{\mu\beta}{}^\nu R_{\rho\sigma\alpha\nu}) - \frac{iq}{2}\gamma^\mu\gamma^\alpha\gamma^\beta f_{\mu\beta}{}^\nu F_{\alpha\nu} \\ &\quad + \frac{i}{16\sqrt{3}}\gamma^\alpha\gamma^\beta\gamma^\rho\gamma^\sigma(3F^\mu{}_\alpha\nabla_\sigma f_{\mu\beta\rho} + F^\mu{}_\sigma\nabla_\mu f_{\alpha\beta\rho} - 2f_{\alpha\beta}{}^\mu\nabla_\mu F_{\rho\sigma}) \\ &\quad - \frac{\sqrt{3}i}{8}\gamma^\rho\gamma^\sigma F^{\mu\nu}\nabla_\sigma f_{\mu\nu\rho}. \end{aligned} \quad (53)$$



To derive the last expression for this commutator, we have made use of the anti-commutativity of Dirac gamma matrices and the following relations

$$\nabla_\mu \gamma^\nu = 0, \quad [\nabla_\mu, \nabla_\nu] = \frac{1}{4} R_{\mu\nu\rho\sigma} \gamma^\rho \gamma^\sigma, \quad f^\rho_{\mu\nu;\rho} = 0. \quad (54)$$

Then we can see that the commutation relation  $[\mathbb{H}_f, \tilde{\mathbb{H}}_D] = 0$  just yields the generalized Killing-Yano equation (52) and the integrability condition for this generalized Killing-Yano tensor of rank-3.

#### IV. SEPARABILITY OF A MASSIVE KLEIN-GORDON COMPLEX SCALAR FIELD EQUATION AND RANK-2 STÄCKEL-KILLING TENSOR

In this section, the massive Klein-Gordon complex scalar field equation with a minimal gauge coupling term is separated by variables into purely radial and purely angular parts in the five-dimensional EMCS-Kerr-AdS black hole geometry. From the separated solutions, we can construct a second-order operator that commutes with the complex scalar Laplacian operator. We then show that this second-order symmetry operator can be compactly expressed in terms of a rank-2 symmetric Stäckel-Killing tensor which has a simple, diagonal form in the chosen local Lorentzian coframe one-forms (8).

To begin with, let us consider a massive Klein-Gordon complex scalar field equation with a minimal electro-magnetic interaction

$$(\square_c - \mu_0^2)\Phi = \frac{1}{\sqrt{-g}}(\partial_\mu + iq\mathcal{A}_\mu)[\sqrt{-g}g^{\mu\nu}(\partial_\nu + iq\mathcal{A}_\nu)\Phi] - \mu_0^2\Phi = 0. \quad (55)$$

The metric determinant for the spacetime (20) is

$$\sqrt{-g} = \frac{rp\Sigma}{(a^2 - b^2)\chi_a\chi_b}, \quad (56)$$

and the contra-invariant components for the metric tensor can be read accordingly from

$$\begin{aligned} g^{\mu\nu}\partial_\mu\partial_\nu &= \eta^{AB}\partial_A \otimes \partial_B \\ &= -\frac{1}{r^4\Delta_r\Sigma} \left[ (r^2 + a^2)(r^2 + b^2)\partial_t + (r^2 + b^2)a\chi_a\partial_\phi + (r^2 + a^2)b\chi_b\partial_\psi \right. \\ &\quad \left. + Q(ab\partial_t + b\chi_a\partial_\phi + a\chi_b\partial_\psi) \right]^2 + \frac{\Delta_r}{\Sigma}\partial_r^2 + \frac{\Delta_p}{\Sigma}\partial_p^2 \\ &\quad + \frac{(p^2 - a^2)^2(p^2 - b^2)^2}{p^4\Delta_p\Sigma} \left( \partial_t - \frac{a\chi_a}{p^2 - a^2}\partial_\phi - \frac{b\chi_b}{p^2 - b^2}\partial_\psi \right)^2 \\ &\quad + \frac{1}{r^2p^2} (ab\partial_t + b\chi_a\partial_\phi + a\chi_b\partial_\psi)^2. \end{aligned} \quad (57)$$

In the background spacetime (20), the massive complex scalar field equation reads

$$\begin{aligned} &\left\{ -\frac{1}{r^4\Delta_r\Sigma} \left[ (r^2 + a^2)(r^2 + b^2)\partial_t + (r^2 + b^2)a\chi_a\partial_\phi + (r^2 + a^2)b\chi_b\partial_\psi \right. \right. \\ &\quad \left. + Q(ab\partial_t + b\chi_a\partial_\phi + a\chi_b\partial_\psi) + \frac{\sqrt{3}}{2}iqQr^2 \right]^2 + \frac{1}{r\Sigma}\partial_r(r\Delta_r\partial_r) \\ &\quad + \frac{1}{p\Sigma}\partial_p(p\Delta_p\partial_p) + \frac{(p^2 - a^2)^2(p^2 - b^2)^2}{p^4\Delta_p\Sigma} \left( \partial_t - \frac{a\chi_a}{p^2 - a^2}\partial_\phi - \frac{b\chi_b}{p^2 - b^2}\partial_\psi \right)^2 \\ &\quad \left. + \frac{1}{r^2p^2} (ab\partial_t + b\chi_a\partial_\phi + a\chi_b\partial_\psi)^2 - \mu_0^2 \right\} \Phi = 0. \end{aligned} \quad (58)$$

With the ansatz of separation of variables  $\Phi = R(r)S(p)e^{i(m\phi + k\psi - \omega t)}$ , it apparently can be separated into a radial

part and an angular part,

$$\begin{aligned} \frac{1}{r} \partial_r (r \Delta_r \partial_r R) + \left\{ \frac{1}{r^4 \Delta_r} \left[ (r^2 + a^2)(r^2 + b^2) \omega - (r^2 + b^2) m a \chi_a \right. \right. \\ \left. \left. - (r^2 + a^2) k b \chi_b + Q (a b \omega - m b \chi_a - k a \chi_b) - \frac{\sqrt{3}}{2} q Q r^2 \right]^2 \right. \\ \left. - \frac{1}{r^2} (a b \omega - m b \chi_a - k a \chi_b)^2 - \mu_0^2 r^2 - \lambda_0^2 \right\} R(r) = 0, \end{aligned} \quad (59)$$

$$\begin{aligned} \frac{1}{p} \partial_p (p \Delta_p \partial_p S) - \left\{ \frac{(p^2 - a^2)^2 (p^2 - b^2)^2}{p^4 \Delta_p} \left( \omega + \frac{m a \chi_a}{p^2 - a^2} + \frac{k b \chi_b}{p^2 - b^2} \right)^2 \right. \\ \left. + \frac{1}{p^2} (a b \omega - m b \chi_a - k a \chi_b)^2 + \mu_0^2 p^2 - \lambda_0^2 \right\} S(p) = 0. \end{aligned} \quad (60)$$

Both of them can be transformed into the general form of Heun equation [25].

Now from the separated parts (59) and (60), one can construct a new dual equation as follows:

$$\begin{aligned} \left\{ - \frac{p^2}{r^4 \Delta_r \Sigma} \left[ (r^2 + a^2)(r^2 + b^2) \partial_t + (r^2 + b^2) a \chi_a \partial_\phi + (r^2 + a^2) b \chi_b \partial_\psi + \frac{\sqrt{3}}{2} i q Q r^2 \right. \right. \\ \left. \left. + Q (a b \partial_t + b \chi_a \partial_\phi + a \chi_b \partial_\psi) \right]^2 + \frac{p^2}{r \Sigma} \partial_r (r \Delta_r \partial_r) - \frac{r^2}{p \Sigma} \partial_p (p \Delta_p \partial_p) \right. \\ \left. - r^2 \frac{(p^2 - a^2)^2 (p^2 - b^2)^2}{p^4 \Delta_p \Sigma} \left( \partial_t - \frac{a \chi_a}{p^2 - a^2} \partial_\phi - \frac{b \chi_b}{p^2 - b^2} \partial_\psi \right)^2 \right. \\ \left. + \frac{p^2 - r^2}{r^2 p^2} (a b \partial_t + b \chi_a \partial_\phi + a \chi_b \partial_\psi)^2 - \lambda_0^2 \right\} \Phi = 0, \end{aligned} \quad (61)$$

from which by setting  $q = 0$  we can extract a second-order symmetric tensor — the so-called Stäckel-Killing tensor

$$\begin{aligned} K^{\mu\nu} \partial_\mu \partial_\nu = & -p^2 \frac{1}{r^4 \Delta_r \Sigma} \left[ (r^2 + a^2)(r^2 + b^2) \partial_t + (r^2 + b^2) a \chi_a \partial_\phi + (r^2 + a^2) b \chi_b \partial_\psi \right. \\ & \left. + Q (a b \partial_t + b \chi_a \partial_\phi + a \chi_b \partial_\psi) \right]^2 + p^2 \frac{\Delta_r}{\Sigma} \partial_r^2 - r^2 \frac{\Delta_p}{\Sigma} \partial_p^2 \\ & - r^2 \frac{(p^2 - a^2)^2 (p^2 - b^2)^2}{p^4 \Delta_p \Sigma} \left( \partial_t - \frac{a \chi_a}{p^2 - a^2} \partial_\phi - \frac{b \chi_b}{p^2 - b^2} \partial_\psi \right)^2 \\ & + \frac{p^2 - r^2}{r^2 p^2} (a b \partial_t + b \chi_a \partial_\phi + a \chi_b \partial_\psi)^2. \end{aligned} \quad (62)$$

This rank-2 tensor has a concise expression given by Eq. (32) in terms of the local Lorentzian orthonormal pentad (8).

Using this Stäckel-Killing tensor, the above dual equation can be put into an operator form

$$(\mathbb{K}_c - \lambda_0^2) \Phi = \frac{1}{\sqrt{-g}} (\partial_\mu + i q \mathcal{A}_\mu) [\sqrt{-g} K^{\mu\nu} (\partial_\nu + i q \mathcal{A}_\nu) \Phi] - \lambda_0^2 \Phi = 0. \quad (63)$$

Clearly, the symmetry operator  $\mathbb{K}_c$  is expressed in terms of the Stäckel-Killing tensor and commutes with the complex scalar Laplacian operator  $\square_c$ . The introduced separation constant  $\lambda_0^2$  acts as the eigenvalue of this operator. Expanding the commutator  $[\mathbb{K}_c, \square_c] = 0$  yields the Killing equation (28) and the integrability condition for the Stäckel-Killing tensor.

## V. SEPARABILITY OF A COMPLEX MASSIVE SPINOR FIELD EQUATION IN THE $D = 5$ CHONG-CVETIČ-LÜ-POPE BLACK HOLE BACKGROUND

In our previous work [10], the modified Dirac equation for spin-1/2 fermions in the three-equal-charge Cvetič-Youm black hole geometry has been decoupled into purely radial and purely angular parts by using a local orthonormal fünfbein (pentad) formalism. In this section, we shall extend that work to the case of a nonzero cosmological constant by showing that our modified Dirac equation is separable by variables in the  $D = 5$  Chong-Cvetič-Lü-Pope black hole background geometry.

### A. F nfbein formalism of spinor field equation

In Ref. [10], it has been shown that in order to separate the field equation for spin-1/2 fermions in a fixed background spacetime subject to the five-dimensional EMCS supergravity theory, an extra counter-term should be supplemented to the usual Dirac equation. The same thing holds true in minimal  $D = 5$  gauged supergravity theory. Incorporating the minimal electro-magnetic coupling interaction, the action of spin-1/2 spinor particles is therefore given by

$$S_f = \frac{i}{2} \int d^5x \sqrt{-g} \bar{\Psi} \left[ \gamma^\mu (\nabla_\mu + iq\mathcal{A}_\mu) + \frac{i}{4\sqrt{3}} \gamma^\mu \gamma^\nu F_{\mu\nu} + \mu_e \right] \Psi, \quad (64)$$

where  $\Psi$  is a complex four-component Dirac spinor,  $\mu_e$  is the rest mass, and  $q$  is the charge of the electron.

Variation of the above action with respect to the spinor field yields the equation of motion

$$(\tilde{\mathbb{H}}_D + \mu_e) \Psi = \left[ \gamma^A e_A^\mu (\partial_\mu + \Gamma_\mu + iq\mathcal{A}_\mu) + \frac{i}{4\sqrt{3}} \gamma^A \gamma^B F_{AB} + \mu_e \right] \Psi = 0, \quad (65)$$

where  $e_A^\mu$  is the f nfbein (pentad), its inverse  $e_\mu^A$  is defined by  $g_{\mu\nu} = \eta_{AB} e_\mu^A e_\nu^B$ ,  $\Gamma_\mu$  is the spinor connection, and  $\gamma^A$ 's are the five-dimensional gamma matrices obeying the anticommutation relations (Clifford algebra)

$$\{\gamma^A, \gamma^B\} \equiv \gamma^A \gamma^B + \gamma^B \gamma^A = 2\eta^{AB}. \quad (66)$$

To our aim, we choose the following explicit representations for the gamma matrices [6]

$$\begin{aligned} \gamma^0 &= i\sigma^1 \otimes I_2, & \gamma^1 &= -\sigma^2 \otimes \sigma^3, & \gamma^2 &= -\sigma^2 \otimes \sigma^1, \\ \gamma^3 &= -\sigma^2 \otimes \sigma^2, & \gamma^5 &= \sigma^3 \otimes I_2 = -i\gamma^0 \gamma^1 \gamma^2 \gamma^3, \end{aligned} \quad (67)$$

where  $\sigma^i$ 's are the Pauli matrices, and  $I_2$  is a  $2 \times 2$  identity matrix, respectively.

In the f nfbein formalism, the modified Dirac field equation (65) can be rewritten in the local Lorentzian frame as [10]

$$(\tilde{\mathbb{H}}_D + \mu_e) \Psi = \left[ \gamma^A (\partial_A + \Gamma_A + iq\mathcal{A}_A) + \frac{i}{4\sqrt{3}} \gamma^A \gamma^B F_{AB} + \mu_e \right] \Psi = 0, \quad (68)$$

where  $\partial_A = e_A^\mu \partial_\mu$  is the local partial differential operator and  $\Gamma_A = e_A^\mu \Gamma_\mu$  is the component of the spinor connection projected in the local Lorentzian frame. Note that the five-dimensional Clifford algebra has two different but reducible representations which can differ by the multiplier of a  $\gamma^5$  matrix. It is usually assumed that fermion fields are in a reducible representation of the Clifford algebra.

### B. Computation of covariant spinor differential operator

In order to get the explicit expression of the modified Dirac's equation in the local Lorentzian frame, one has to find firstly the local partial differential operator  $\partial_A = e_A^\mu \partial_\mu$  and the spinor connection  $\Gamma_A = e_A^\mu \Gamma_\mu$  subject to the background metric (20). Once the pentad coframe one-forms  $e^A = e_\mu^A dx^\mu$  have been concretely chosen, the local differential operator  $\partial_A = e_A^\mu \partial_\mu$  can be determined via the orthogonal relations:  $e_A^\mu e_\mu^B = \delta_A^B$  and  $e_A^\mu e_\nu^A = \delta_\nu^\mu$ .

The orthonormal basis one-vectors  $\partial_A$  dual to the pentad  $e^A$  given in Eq. (8) are given by

$$\begin{aligned} \partial_0 &= \frac{1}{r^2 \sqrt{\Delta_r \Sigma}} \left[ (r^2 + a^2)(r^2 + b^2) \partial_t + (r^2 + b^2) a \chi_a \partial_\phi \right. \\ &\quad \left. + (r^2 + a^2) b \chi_b \partial_\psi + Q(ab \partial_t + b \chi_a \partial_\phi + a \chi_b \partial_\psi) \right], \\ \partial_1 &= \sqrt{\frac{\Delta_r}{\Sigma}} \partial_r, & \partial_2 &= \sqrt{\frac{\Delta_p}{\Sigma}} \partial_p, \\ \partial_3 &= \frac{(p^2 - a^2)(p^2 - b^2)}{p^2 \sqrt{\Delta_p \Sigma}} \left( \partial_t - \frac{a \chi_a}{p^2 - a^2} \partial_\phi - \frac{b \chi_b}{p^2 - b^2} \partial_\psi \right), \\ \partial_5 &= \frac{1}{rp} (ab \partial_t + b \chi_a \partial_\phi + a \chi_b \partial_\psi). \end{aligned} \quad (69)$$

Therefore, the spinor partial differential operator is

$$\begin{aligned}
\gamma^A \partial_A = & \gamma^0 \frac{1}{r^2 \sqrt{\Delta_r \Sigma}} \left[ (r^2 + a^2)(r^2 + b^2) \partial_t + (r^2 + b^2) a \chi_a \partial_\phi + (r^2 + a^2) b \chi_b \partial_\psi \right. \\
& + Q(ab \partial_t + b \chi_a \partial_\phi + a \chi_b \partial_\psi) \left. \right] + \gamma^1 \sqrt{\frac{\Delta_r}{\Sigma}} \partial_r + \gamma^2 \sqrt{\frac{\Delta_p}{\Sigma}} \partial_p \\
& + \gamma^3 \frac{(p^2 - a^2)(p^2 - b^2)}{p^2 \sqrt{\Delta_p \Sigma}} \left( \partial_t - \frac{a \chi_a}{p^2 - a^2} \partial_\phi - \frac{b \chi_b}{p^2 - b^2} \partial_\psi \right) \\
& + \gamma^5 \frac{1}{rp} (ab \partial_t + b \chi_a \partial_\phi + a \chi_b \partial_\psi), \tag{70}
\end{aligned}$$

The next step is to compute the component  $\Gamma_A$  of the spinor connection. The procedure of the detailed computations is outlined in the appendix. For our purpose, we need the final expression

$$\begin{aligned}
\gamma^A \Gamma_A = & \gamma^1 \sqrt{\frac{\Delta_r}{\Sigma}} \left( \frac{\Delta'_r}{4\Delta_r} + \frac{1}{2r} + \frac{r - ip\gamma^5}{2\Sigma} \right) + \gamma^2 \sqrt{\frac{\Delta_p}{\Sigma}} \left( \frac{\Delta'_p}{4\Delta_p} + \frac{1}{2p} + \frac{p + ir\gamma^5}{2\Sigma} \right) \\
& + \left( \frac{Q + ab}{2r^2\Sigma} + \frac{ab}{2p^2\Sigma} \right) i\gamma^0 \gamma^1 (r + ip\gamma^5) - \frac{Q}{2\Sigma^2} (ir\gamma^0 \gamma^1 + p\gamma^0 \gamma^1 \gamma^5), \tag{71}
\end{aligned}$$

where a prime denotes the partial differential with respect to the coordinates  $r$  and  $p$ .

As explained out in [10], the appearance of last term in the expression of  $\gamma^A \Gamma_A$  spoils the separability of the usual Dirac equation. To cancel it, one should supplement an additional term

$$\frac{i}{4\sqrt{3}} \gamma^A \gamma^B F_{AB} = \frac{iQ}{2\Sigma^2} (r\gamma^0 \gamma^1 - p\gamma^2 \gamma^3) \equiv \frac{Q}{2\Sigma^2} (p\gamma^0 \gamma^1 + r\gamma^2 \gamma^3) \gamma^5 = \frac{1}{12\sqrt{3}} \gamma^A \gamma^B \gamma^C \tilde{F}_{ABC}. \tag{72}$$

With the inclusion of this new counter-term, then the modified Dirac equation for a spin-1/2 spinor field in this general rotating, charged, EMCS-Kerr-AdS black hole spacetime can be completely decoupled into purely radial and purely angular parts. Without this counter-term, the usual Dirac equation is only separable [14] in the special case when  $a = \pm b$ . It should be emphasized that the work on the separation of Dirac equation in [14] is incomplete, since our counter-term still make a contribution to the Dirac equation in the  $a = \pm b$  case.

Combining the above expressions with the minimal gauge coupling factor

$$iq\gamma^\mu \mathcal{A}_\mu = iq\gamma^A \mathcal{A}_A = \gamma^0 \frac{\sqrt{3}iqQ}{2\sqrt{\Delta_r \Sigma}}, \tag{73}$$

we find that the modified Dirac's covariant differential operator in the local Lorentzian frame is

$$\begin{aligned}
\tilde{\mathbb{H}}_D = & \gamma^0 \frac{1}{r^2 \sqrt{\Delta_r \Sigma}} \left[ (r^2 + a^2)(r^2 + b^2) \partial_t + (r^2 + b^2) a \chi_a \partial_\phi + (r^2 + a^2) b \chi_b \partial_\psi \right. \\
& + Q(ab \partial_t + b \chi_a \partial_\phi + a \chi_b \partial_\psi) + \frac{\sqrt{3}}{2} iqQr^2 \left. \right] + \gamma^1 \sqrt{\frac{\Delta_r}{\Sigma}} \left( \partial_r + \frac{\Delta'_r}{4\Delta_r} + \frac{1}{2r} + \frac{r - ip\gamma^5}{2\Sigma} \right) \\
& + \gamma^2 \sqrt{\frac{\Delta_p}{\Sigma}} \left[ \partial_p + \frac{\Delta'_p}{4\Delta_p} + \frac{1}{2p} + \frac{i\gamma^5}{2\Sigma} (r - ip\gamma^5) \right] + \gamma^3 \frac{(p^2 - a^2)(p^2 - b^2)}{p^2 \sqrt{\Delta_p \Sigma}} \left( \partial_t - \frac{a \chi_a}{p^2 - a^2} \partial_\phi \right. \\
& \left. - \frac{b \chi_b}{p^2 - b^2} \partial_\psi \right) + \gamma^5 \frac{1}{rp} (ab \partial_t + b \chi_a \partial_\phi + a \chi_b \partial_\psi) + \left( \frac{Q + ab}{2r^2\Sigma} + \frac{ab}{2p^2\Sigma} \right) i\gamma^0 \gamma^1 (r + ip\gamma^5). \tag{74}
\end{aligned}$$

### C. Separation of variables of spinor field equation

With the above preparation in hand, we are ready to decouple the modified Dirac equation

$$(\tilde{\mathbb{H}}_D + \mu_e) \Psi = \left[ \gamma^\mu (\partial_\mu + \Gamma_\mu + iq\mathcal{A}_\mu) + \frac{i}{4\sqrt{3}} \gamma^\mu \gamma^\nu F_{\mu\nu} + \mu_e \right] \Psi = 0. \tag{75}$$

Substituting the above spinor covariant differential operator into Eq. (75) and multiplying it a factor  $(r - ip\gamma^5)\sqrt{r + ip\gamma^5} = \sqrt{\Sigma(r - ip\gamma^5)}$  by the left, then after some lengthy algebra manipulations we get the following expression for the modified Dirac equation in the five-dimensional EMCS-Kerr-AdS metric

$$\begin{aligned} & \left\{ \gamma^0 \frac{1}{r^2 \sqrt{\Delta_r}} \left[ (r^2 + a^2)(r^2 + b^2) \partial_t + (r^2 + b^2) a \chi_a \partial_\phi + (r^2 + a^2) b \chi_b \partial_\psi \right. \right. \\ & \quad \left. \left. + Q(ab \partial_t + b \chi_a \partial_\phi + a \chi_b \partial_\psi) + \frac{\sqrt{3}}{2} i q Q r^2 \right] + \gamma^1 \sqrt{\Delta_r} \left( \partial_r + \frac{\Delta'_r}{4 \Delta_r} + \frac{1}{2r} \right) \right. \\ & \quad \left. + \gamma^2 \sqrt{\Delta_p} \left( \partial_p + \frac{\Delta'_p}{4 \Delta_p} + \frac{1}{2p} \right) + \gamma^3 \frac{(p^2 - a^2)(p^2 - b^2)}{p^2 \sqrt{\Delta_p}} \left( \partial_t - \frac{a \chi_a}{p^2 - a^2} \partial_\phi - \frac{b \chi_b}{p^2 - b^2} \partial_\psi \right) \right. \\ & \quad \left. + \left( \frac{\gamma^5}{p} - \frac{i}{r} \right) (ab \partial_t + b \chi_a \partial_\phi + a \chi_b \partial_\psi) + \left( \frac{Q + ab}{2r^2} + \frac{ab}{2p^2} \right) i \gamma^0 \gamma^1 \right. \\ & \quad \left. + \mu_e (r - ip\gamma^5) \right\} (\sqrt{r + ip\gamma^5} \Psi) = 0. \end{aligned} \quad (76)$$

Now we assume that the spin-1/2 fermion fields are in a reducible representation of the Clifford algebra and can be taken as a complex four-component Dirac spinor. Adopting the explicit representation (67) for the gamma matrices and the following ansatz for the separation of variables

$$\sqrt{r + ip\gamma^5} \Psi = e^{i(m\phi + k\psi - \omega t)} \begin{pmatrix} R_2(r) S_1(p) \\ R_1(r) S_2(p) \\ R_1(r) S_1(p) \\ R_2(r) S_2(p) \end{pmatrix}, \quad (77)$$

we find that the modified Dirac equation in the five-dimensional EMCS-Kerr-AdS metric can be decoupled into the purely radial parts and the purely angular parts

$$\sqrt{\Delta_r} \mathcal{D}_r^- R_1 = \left[ \lambda + i\mu_e r - \frac{Q + ab}{2r^2} - \frac{i}{r} (ab\omega - mb\chi_a - ka\chi_b) \right] R_2, \quad (78)$$

$$\sqrt{\Delta_r} \mathcal{D}_r^+ R_2 = \left[ \lambda - i\mu_e r - \frac{Q + ab}{2r^2} + \frac{i}{r} (ab\omega - mb\chi_a - ka\chi_b) \right] R_1, \quad (79)$$

$$\sqrt{\Delta_p} \mathcal{L}_p^+ S_1 = \left[ \lambda + \mu_e p + \frac{ab}{2p^2} + \frac{1}{p} (ab\omega - mb\chi_a - ka\chi_b) \right] S_2, \quad (80)$$

$$\sqrt{\Delta_p} \mathcal{L}_p^- S_2 = \left[ -\lambda + \mu_e p - \frac{ab}{2p^2} + \frac{1}{p} (ab\omega - mb\chi_a - ka\chi_b) \right] S_1, \quad (81)$$

in which  $\lambda$  is the separation constant, and we have introduced

$$\begin{aligned} \mathcal{D}_r^\pm &= \partial_r + \frac{\Delta'_r}{4 \Delta_r} + \frac{1}{2r} \pm i \frac{1}{r^2 \Delta_r} \left[ (r^2 + a^2)(r^2 + b^2) \omega - (r^2 + b^2) m a \chi_a \right. \\ & \quad \left. - (r^2 + a^2) k b \chi_b + Q(ab\omega - mb\chi_a - ka\chi_b) - \frac{\sqrt{3}}{2} q Q r^2 \right], \\ \mathcal{L}_p^\pm &= \partial_p + \frac{\Delta'_p}{4 \Delta_p} + \frac{1}{2p} \pm \frac{(p^2 - a^2)(p^2 - b^2)}{p^2 \Delta_p} \left( \omega + \frac{m a \chi_a}{p^2 - a^2} + \frac{k b \chi_b}{p^2 - b^2} \right). \end{aligned}$$

The separated radial and angular equations (78-81) can be reduced into a master equation containing only one component. The decoupled master equations are very complicated and not given here. As for the exact solution to these equations, one hopes that they can be transformed into the general form of Heun equation [25]. Besides, the angular part can be transformed into the radial part if one replaces  $p$  by  $ir$  in the vacuum case when  $M = Q = 0$ .

## VI. SEPARABILITY OF DUAL FIRST-ORDER DIFFERENTIAL OPERATOR EQUATION IN TERMS OF GENERALIZED KILLING-YANO TENSOR

In the last section, we have explicitly shown that the modified Dirac's equation is separable in the  $D = 5$  EMCS-Kerr-AdS black hole spacetime. In this section, we will demonstrate that this separability is closely related to the

existence of a rank-3 generalized Killing-Yano tensor admitted by this spacetime metric. To this end, we will show that the separation constant introduced in the separated radial and angular parts of the modified Dirac equation acts as the eigenvalue of the first-order differential operator  $\mathbb{H}_f$

$$(\mathbb{H}_f + \lambda)\Psi = 0, \quad (82)$$

which implies that it commutes with the modified Dirac operator.

This symmetry operator is expressed in terms of the rank-3 generalized Killing-Yano tensor and explicitly given by Eq. (45). Using an identity  $f_{\mu\nu;\rho}^\rho = 0$  and the definition  $W_{\mu\nu\rho\sigma} = -f_{\mu\nu\rho;\sigma} + f_{\nu\rho\sigma;\mu} - f_{\rho\sigma\mu;\nu} + f_{\sigma\mu\nu;\rho}$  as well as the anti-commutative property of gamma matrices, one can also write this operator in another equivalent form

$$\mathbb{H}_f = -\frac{1}{2}\gamma^\mu\gamma^\nu f_{\mu\nu}^\rho(\partial_\rho + \Gamma_\rho + iq\mathcal{A}_\rho) - \frac{1}{64}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma W_{\mu\nu\rho\sigma}. \quad (83)$$

We now proceed to write out the explicit expression of this operator in the 5-dimensional EMCS-Kerr-AdS black hole background. We first compute the term  $-\frac{1}{2}\gamma^\mu\gamma^\nu f_{\mu\nu}^\rho(\nabla_\rho + iq\mathcal{A}_\rho) = -\frac{1}{2}\gamma^\mu\gamma^\nu f_{\mu\nu}^\rho(\partial_\rho + \Gamma_\rho + iq\mathcal{A}_\rho)$ . After some tedious algebra manipulations, we get

$$\begin{aligned} & -\frac{1}{2}\gamma^\mu\gamma^\nu f_{\mu\nu}^\rho(\partial_\rho + \Gamma_\rho + iq\mathcal{A}_\rho) \\ &= \gamma^5\gamma^0 p \sqrt{\frac{\Delta_r}{\Sigma}} \left( \partial_r + \frac{\Delta'_r}{4\Delta_r} + \frac{1}{2r} + \frac{r}{2\Sigma} \right) + \gamma^5\gamma^1 p \frac{1}{r^2\sqrt{\Delta_r\Sigma}} \left[ (r^2 + a^2)(r^2 + b^2)\partial_t \right. \\ & \quad \left. + (r^2 + b^2)a\chi_a\partial_\phi + (r^2 + a^2)b\chi_b\partial_\psi + Q(ab\partial_t + b\chi_a\partial_\phi + a\chi_b\partial_\psi) + \frac{\sqrt{3}}{2}iqQr^2 \right] \\ & \quad + \gamma^5\gamma^2(-r) \frac{(p^2 - a^2)(p^2 - b^2)}{p^2\sqrt{\Delta_p\Sigma}} \left( \partial_t - \frac{a\chi_a}{p^2 - a^2}\partial_\phi - \frac{b\chi_b}{p^2 - b^2}\partial_\psi \right) + \gamma^5\gamma^3 r \sqrt{\frac{\Delta_p}{\Sigma}} \left( \partial_p + \frac{\Delta'_p}{4\Delta_p} \right. \\ & \quad \left. + \frac{1}{2p} + \frac{p}{2\Sigma} \right) + (p\gamma^0\gamma^1 - r\gamma^2\gamma^3) \frac{1}{rp} (ab\partial_t + b\chi_a\partial_\phi + a\chi_b\partial_\psi) + \frac{Q}{2\Sigma} - \frac{Q + ab}{2r^2} + \frac{ab}{2p^2} \\ & \quad - \left( \frac{ab}{rp} + \frac{Qp}{r\Sigma} \right) i\gamma^5 + i\sqrt{\frac{\Delta_r}{\Sigma}} \left( \frac{p^2}{2\Sigma} - \frac{3}{2} \right) \gamma^0 + i\sqrt{\frac{\Delta_p}{\Sigma}} \left( \frac{3}{2} - \frac{r^2}{2\Sigma} \right) \gamma^3. \end{aligned} \quad (84)$$

Next, we consider the last term in the operator  $\mathbb{H}_f$ . This can be easily done by considering the exterior differential of the rank-3 generalized Killing-Yano tensor

$$W = df = -4\left(\frac{ab}{rp} + \frac{Qp}{r\Sigma}\right) e^0 \wedge e^1 \wedge e^2 \wedge e^3 - 4\sqrt{\frac{\Delta_p}{\Sigma}} e^0 \wedge e^1 \wedge e^2 \wedge e^5 + 4\sqrt{\frac{\Delta_r}{\Sigma}} e^1 \wedge e^2 \wedge e^3 \wedge e^5, \quad (85)$$

with which we obtain the counter-term

$$-\frac{1}{64}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma W_{\mu\nu\rho\sigma} = \frac{3i}{2} \left[ \gamma^0 \sqrt{\frac{\Delta_r}{\Sigma}} - \gamma^3 \sqrt{\frac{\Delta_p}{\Sigma}} + \gamma^5 \left( \frac{ab}{rp} + \frac{Qp}{r\Sigma} \right) \right], \quad (86)$$

that can exactly cancel the last three unexpected terms in Eq. (84).

Adding these two terms together, the dual equation (82) for the first-order symmetry operator now reads

$$\begin{aligned} & \left\{ \gamma^5\gamma^0 p \sqrt{\frac{\Delta_r}{\Sigma}} \left( \partial_r + \frac{\Delta'_r}{4\Delta_r} + \frac{1}{2r} + \frac{r - ip\gamma^5}{2\Sigma} \right) + \gamma^5\gamma^1 p \frac{1}{r^2\sqrt{\Delta_r\Sigma}} \left[ (r^2 + a^2)(r^2 + b^2)\partial_t \right. \right. \\ & \quad \left. \left. + (r^2 + b^2)a\chi_a\partial_\phi + (r^2 + a^2)b\chi_b\partial_\psi + Q(ab\partial_t + b\chi_a\partial_\phi + a\chi_b\partial_\psi) + \frac{\sqrt{3}}{2}iqQr^2 \right] \right. \\ & \quad \left. + \gamma^5\gamma^2(-r) \frac{(p^2 - a^2)(p^2 - b^2)}{p^2\sqrt{\Delta_p\Sigma}} \left( \partial_t - \frac{a\chi_a}{p^2 - a^2}\partial_\phi - \frac{b\chi_b}{p^2 - b^2}\partial_\psi \right) \right. \\ & \quad \left. + \gamma^5\gamma^3 r \sqrt{\frac{\Delta_p}{\Sigma}} \left( \partial_p + \frac{\Delta'_p}{4\Delta_p} + \frac{1}{2p} + \frac{p + ir\gamma^5}{2\Sigma} \right) + (p\gamma^0\gamma^1 - r\gamma^2\gamma^3) \frac{1}{rp} (ab\partial_t + b\chi_a\partial_\phi \right. \\ & \quad \left. + a\chi_b\partial_\psi) + \frac{iab}{2rp}\gamma^5 + \frac{iQp}{2r\Sigma}\gamma^5 + \frac{Q}{2\Sigma} - \frac{Q + ab}{2r^2} + \frac{ab}{2p^2} + \lambda \right\} \Psi = 0. \end{aligned} \quad (87)$$

To decouple this equation, we multiply it a factor  $(r - i\gamma^5 p)\sqrt{r + i\gamma^5 p}\gamma^5$  from the left and arrive at

$$\begin{aligned} & \left\{ \gamma^0 p \sqrt{\Delta_r} \left( \partial_r + \frac{\Delta'_r}{4\Delta_r} + \frac{1}{2r} \right) + \gamma^1 p \frac{1}{r^2 \sqrt{\Delta_r}} \left[ (r^2 + a^2)(r^2 + b^2) \partial_t + (r^2 + b^2) a \chi_a \partial_\phi \right. \right. \\ & \quad \left. \left. + (r^2 + a^2) b \chi_b \partial_\psi + Q(ab \partial_t + b \chi_a \partial_\phi + a \chi_b \partial_\psi) + \frac{\sqrt{3}}{2} i q Q r^2 \right] \right. \\ & \quad \left. + \gamma^2 (-r) \frac{(p^2 - a^2)(p^2 - b^2)}{p^2 \sqrt{\Delta_p}} \left( \partial_t - \frac{a \chi_a}{p^2 - a^2} \partial_\phi - \frac{b \chi_b}{p^2 - b^2} \partial_\psi \right) \right. \\ & \quad \left. + \gamma^3 r \sqrt{\Delta_p} \left( \partial_p + \frac{\Delta'_p}{4\Delta_p} + \frac{1}{2p} \right) - i \gamma^0 \gamma^1 \frac{\Sigma}{r p} (ab \partial_t + b \chi_a \partial_\phi + a \chi_b \partial_\psi) \right. \\ & \quad \left. + \frac{i(Q + ab)p}{2r^2} + \frac{abr}{2p^2} \gamma^5 + \lambda(\gamma^5 r - ip) \right\} (\sqrt{r + ip\gamma^5} \Psi) = 0. \end{aligned} \quad (88)$$

After using the relation  $i\gamma^5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3$ , the above equation can be split as

$$\begin{aligned} & \left\{ \gamma^0 \frac{1}{r^2 \sqrt{\Delta_r}} \left[ (r^2 + a^2)(r^2 + b^2) \partial_t + (r^2 + b^2) a \chi_a \partial_\phi + (r^2 + a^2) b \chi_b \partial_\psi + \frac{\sqrt{3}}{2} i q Q r^2 \right. \right. \\ & \quad \left. \left. + Q(ab \partial_t + b \chi_a \partial_\phi + a \chi_b \partial_\psi) \right] + \gamma^1 \sqrt{\Delta_r} \left( \partial_r + \frac{\Delta'_r}{4\Delta_r} + \frac{1}{2r} \right) - \frac{i}{r} (ab \partial_t + b \chi_a \partial_\phi \right. \\ & \quad \left. + a \chi_b \partial_\psi) + \frac{Q + ab}{2r^2} i \gamma^0 \gamma^1 + \mu_e r - i \lambda \gamma^0 \gamma^1 \right\} (\sqrt{r + ip\gamma^5} \Psi) = 0, \end{aligned} \quad (89)$$

$$\begin{aligned} & \left\{ \gamma^2 \sqrt{\Delta_p} \left( \partial_p + \frac{\Delta'_p}{4\Delta_p} + \frac{1}{2p} \right) + \gamma^3 \frac{(p^2 - a^2)(p^2 - b^2)}{p^2 \sqrt{\Delta_p}} \left( \partial_t - \frac{a \chi_a}{p^2 - a^2} \partial_\phi - \frac{b \chi_b}{p^2 - b^2} \partial_\psi \right) \right. \\ & \quad \left. + \frac{\gamma^5}{p} (ab \partial_t + b \chi_a \partial_\phi + a \chi_b \partial_\psi) + \frac{iab}{2p^2} \gamma^0 \gamma^1 - i \mu_e p \gamma^5 + i \lambda \gamma^0 \gamma^1 \right\} (\sqrt{r + ip\gamma^5} \Psi) = 0, \end{aligned} \quad (90)$$

which reduce to the separated radial and angular equations (78-81) when the explicit representations (67) for gamma matrices have been used.

From this, we can see that  $\tilde{\mathbb{H}}_D \mathbb{H}_f \Psi = \mu_e \lambda \Psi = \mathbb{H}_f \tilde{\mathbb{H}}_D \Psi$  and draw a conclusion that it is the existence of a rank-3 generalized Killing-Yano tensor that ensures the separability of the modified Dirac equation in the five-dimensional EMCS-Kerr-AdS black hole geometry.

The first-order symmetry operator  $\mathbb{H}_f$  can be thought of as the “square root” of the second-order operator  $\mathbb{K}_c$ . It has a lot of correspondences in different contexts. It is a five-dimensional analogue to the nonstandard Dirac operator discovered by Carter and McLenaghan [4] for the four-dimensional Kerr metric. This operator corresponds to the nongeneric supersymmetric generator in pseudo-classical mechanics [26].

## VII. CONCLUSIONS

In this paper, the hidden symmetries of the general, non-extremal, rotating, charged, Chong-Cvetič-Lü-Pope [13] (EMCS-Kerr-AdS) black holes in minimal five-dimensional gauged supergravity theory have been completely studied. In particular, we have shown that the existence of the Stäckel-Killing tensor ensures the separation of variables in a massive Klein-Gordon complex scalar field equation, and the separability of a modified Dirac’s equation in this spacetime background is also closely associated with the existence of a generalized Killing-Yano tensor of rank-3.

There are a number of novel characters of this paper. First of all, the whole discussions of our work are elegant and concise because they are based upon a local Lorentzian orthonormal pentad that we have set up for the metric. A special advantage of this is that both the Stäckel-Killing tensor and the generalized (conformal) Killing-Yano tensors have simple expressions within the pentad formalism. Next, we have, for the first time, proposed a suitable generalization of the concepts of (conformal) Killing-Yano tensors so that they can be subject to minimal  $D = 5$  gauged supergravity theory. In our generalization, there is no need to identify the dual Maxwell three-form with the auxiliary “torsion” field. Third, we have constructed two new symmetry operators that, respectively, commute with the complex scalar Laplacian operator and the modified Dirac operator. Finally, we have extended thermodynamics to the case of a variable cosmological constant.

There are also many possible applications of the present work, which can serve as a basis to study various aspects of the spin-1/2 spinor field, such as Hawking radiation, quasinormal modes, absorption rate, perturbation instability,

supersymmetry, nongeneric supersymmetric pseudo-classical mechanics, etc. An interesting and open question is to investigate whether the present work can be applied or slightly extended to more general five-dimensional Cvetič-Youm black holes [11] with three different charges and two independent angular momenta.

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### Appendix: Connection one-forms and Weyl curvature two-forms

In this appendix, we outline the computation process for the spin-connection one-forms, the spinor-connection one-forms, and the curvature two-forms within the fünfbein formalism.

Once the pentad coframe one-forms  $e^A = e^A_\mu dx^\mu$  given in Eq. (8) have been chosen, we first need to figure out their exterior differentials, and then use the torsion-free condition — the Cartan's first structure equation and the skew-symmetric condition

$$de^A + \omega^A_B \wedge e^B = 0, \quad \omega_{AB} = \eta_{AC} \omega^C_B = -\omega_{BA}, \quad (\text{A1})$$

to uniquely determine the spin-connection one-forms  $\omega^A_B = \omega^A_{B\mu} dx^\mu = \Upsilon^A_{BC} e^C$  in the orthonormal pentad coframe.

In order to obtain the spinor connection one-forms  $\Gamma = \Gamma_\mu dx^\mu \equiv \Gamma_A e^A$  from the spin connection one-forms  $\omega_{AB} = \omega_{AB\mu} dx^\mu \equiv \Upsilon_{ABC} e^C$ , one can utilize the homomorphism between the  $\text{SO}(4,1)$  group and its spinor representation derivable from the Clifford algebra (66). The  $\mathfrak{so}(4,1)$  Lie algebra is defined by the ten antisymmetric generators  $\Sigma^{AB} = [\gamma^A, \gamma^B]/(2i)$  which gives the spinor representation, and the spinor connection  $\Gamma$  can be regarded as a  $\mathfrak{so}(4,1)$  Lie-algebra-valued one-form. Using the isomorphism between the  $\mathfrak{so}(4,1)$  Lie algebra and its spinor representation, i.e.,  $\Gamma_\mu = (i/4) \Sigma^{AB} \omega_{AB\mu} = (1/8) [\gamma^A, \gamma^B] \omega_{AB\mu} = (1/4) \gamma^A \gamma^B \omega_{AB\mu}$ , one can immediately construct the spinor connection one-forms

$$\Gamma = \frac{1}{8} [\gamma^A, \gamma^B] \omega_{AB} = \frac{1}{4} \gamma^A \gamma^B \omega_{AB} = \frac{1}{4} \gamma^A \gamma^B \Upsilon_{ABC} e^C, \quad (\text{A2})$$

where  $\Gamma_A = e^\mu_A \Gamma_\mu = (1/4) \gamma^B \gamma^C \Upsilon_{BCA}$  is the component of the spinor connection in the local Lorentzian frame.

At the last step, we can easily read  $\Gamma_A$  from the spinor connection one-forms  $\Gamma \equiv \Gamma_A e^A = (1/4) \gamma^A \gamma^B \omega_{AB}$ . The explicit expressions for the spin-connection one-forms and the five pentad components of the spinor-connection one-forms are presented in Eqs. (A2) and (A3) of [10], where  $\Delta_r$  and  $\Delta_p$  should be replaced by the corresponding expressions given in this paper (subject to the case with a nonzero cosmological constant). We refer the reader to the appendix of our previous paper [10].

Taking use of the properties of gamma matrices together with the relation  $\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$ , we get

$$\begin{aligned} \gamma^A \Gamma_A &= \frac{1}{4} \gamma^A \gamma^B \gamma^C \Upsilon_{BCA} \\ &= \gamma^1 \sqrt{\frac{\Delta_r}{\Sigma}} \left( \frac{\Delta'_r}{4\Delta_r} + \frac{1}{2r} + \frac{r - ip\gamma^5}{2\Sigma} \right) + \gamma^2 \sqrt{\frac{\Delta_p}{\Sigma}} \left( \frac{\Delta'_p}{4\Delta_p} + \frac{1}{2p} + \frac{p + ir\gamma^5}{2\Sigma} \right) \\ &\quad + \left( \frac{ab}{2r^2 p^2} + \frac{Q}{2r^2 \Sigma} \right) i\gamma^0 \gamma^1 (r + ip\gamma^5) - \frac{Q}{2\Sigma^2} i\gamma^0 \gamma^1 (r - ip\gamma^5), \end{aligned} \quad (\text{A3})$$

where a prime denotes the partial differential with respect to the coordinates  $r$  and  $p$ .

Using our pentad (8), the Riemann curvature two-forms  $\mathcal{R}^A_B = d\omega^A_B + \omega^A_C \wedge \omega^C_B$  can be concisely expressed by Eq. (A5) in the appendix of Ref. [10], where one has to add a term  $-1/l^2$  into the expressions of the coefficients  $(\alpha, \beta, \gamma, \delta, \varepsilon)$ , while  $(C_0, C_1, C_2, C_3, C_4)$  remain formally unchanged and are given below.

The Ricci tensors, the scalar curvature, and the Einstein tensors for the  $D = 5$  EMCS-Kerr-AdS metric are

$$-R_{00} = R_{11} = -\frac{4}{l^2} - \frac{2Q^2(2r^2 + p^2)}{\Sigma^4}, \quad R_{22} = R_{33} = -\frac{4}{l^2} + \frac{2Q^2(r^2 + 2p^2)}{\Sigma^4}, \quad (\text{A4})$$

$$R_{55} = -\frac{4}{l^2} + \frac{2Q^2(r^2 - p^2)}{\Sigma^4}, \quad R = -\frac{20}{l^2} - \frac{2Q^2(r^2 - p^2)}{\Sigma^4}. \quad (\text{A5})$$

$$-G_{00} = G_{11} = \frac{6}{l^2} - \frac{3Q^2}{\Sigma^3}, \quad G_{22} = G_{33} = \frac{6}{l^2} + \frac{3Q^2}{\Sigma^3}, \quad G_{55} = \frac{6}{l^2} + \frac{3Q^2(r^2 - p^2)}{\Sigma^4}. \quad (\text{A6})$$

Using the pentad (8), the  $U(1)$  gauge potential one-form can be written as

$$\mathcal{A} = \frac{\sqrt{3}Q}{2\sqrt{\Delta_r \Sigma}} e^0, \quad (\text{A7})$$



the field strength two-form and its corresponding Hodge dual three-form are

$$F = dA = \frac{\sqrt{3}Q}{\Sigma^2} (r e^0 \wedge e^1 - p e^2 \wedge e^3), \quad (\text{A8})$$

$$\tilde{F} = *F = \frac{\sqrt{3}Q}{\Sigma^2} (p e^0 \wedge e^1 + r e^2 \wedge e^3) \wedge e^5. \quad (\text{A9})$$

The complete Einstein equations are satisfied by the energy-momentum tensor of  $U(1)$  gauge field

$$-T_{00} = T_{11} = -\frac{3Q^2}{2\Sigma^3}, \quad T_{22} = T_{33} = +\frac{3Q^2}{2\Sigma^3}, \quad T_{55} = \frac{3Q^2(r^2 - p^2)}{2\Sigma^4}. \quad (\text{A10})$$

The Maxwell-Chern-Simons equation can be rewritten as

$$\partial_\nu (\sqrt{-g} F^{\mu\nu}) + \frac{1}{2\sqrt{3}} \epsilon^{\mu\nu\alpha\beta\gamma} F_{\nu\alpha} F_{\beta\gamma} = 0, \quad (\text{A11})$$

and is satisfied by verifying that

$$d\tilde{F} = -\frac{4\sqrt{3}Q^2 rp}{\Sigma^4} e^0 \wedge e^1 \wedge e^2 \wedge e^3 = \frac{2}{\sqrt{3}} F \wedge F. \quad (\text{A12})$$

Finally, we present the explicit expression for the Weyl curvature two-forms in the pentad formalism as follows:

$$\begin{aligned} \mathcal{C}_1^0 &= A e^0 \wedge e^1 + 2C_1 e^1 \wedge e^5 - 2C_0 e^2 \wedge e^3 + 2C_2 e^2 \wedge e^5, \\ \mathcal{C}_2^0 &= B e^0 \wedge e^2 - C_0 e^1 \wedge e^3 + C_2 e^1 \wedge e^5 - C_1 e^2 \wedge e^5, \\ \mathcal{C}_3^0 &= B e^0 \wedge e^3 - C_3 e^0 \wedge e^5 + C_0 e^1 \wedge e^2 - C_1 e^3 \wedge e^5, \\ \mathcal{C}_5^0 &= -C_3 e^0 \wedge e^3 + C e^0 \wedge e^5 - C_2 e^1 \wedge e^2, \\ \mathcal{C}_{12}^0 &= -C_0 e^0 \wedge e^3 + C_2 e^0 \wedge e^5 + B e^1 \wedge e^2 - C_4 e^3 \wedge e^5, \\ \mathcal{C}_3^1 &= C_0 e^0 \wedge e^2 + B e^1 \wedge e^3 - C_3 e^1 \wedge e^5 + C_4 e^2 \wedge e^5, \\ \mathcal{C}_5^1 &= -2C_1 e^0 \wedge e^1 - C_2 e^0 \wedge e^2 - C_3 e^1 \wedge e^3 + C e^1 \wedge e^5 + 2C_4 e^2 \wedge e^3, \\ \mathcal{C}_3^2 &= 2C_0 e^0 \wedge e^1 + 2C_4 e^1 \wedge e^5 + D e^2 \wedge e^3 + 2C_3 e^2 \wedge e^5, \\ \mathcal{C}_5^2 &= -2C_2 e^0 \wedge e^1 + C_1 e^0 \wedge e^2 + C_4 e^1 \wedge e^3 + 2C_3 e^2 \wedge e^3 - C e^2 \wedge e^5, \\ \mathcal{C}_5^3 &= C_1 e^0 \wedge e^3 - C_4 e^1 \wedge e^2 - C e^3 \wedge e^5, \end{aligned} \quad (\text{A13})$$

where

$$\begin{aligned} A &= \frac{2M}{\Sigma^3} (3r^2 - p^2) - \frac{8Qab}{\Sigma^3} - \frac{Q^2(15r^2 + 11p^2)}{2\Sigma^4}, \\ B &= -\frac{2M}{\Sigma^3} (r^2 - p^2) + \frac{4Qab}{\Sigma^3} + \frac{Q^2(5r^2 + 3p^2)}{2\Sigma^4}, \\ C &= -\frac{2M}{\Sigma^2} + \frac{5Q^2}{2\Sigma^3}, \quad D = \frac{2M}{\Sigma^3} (r^2 - 3p^2) - \frac{8Qab}{\Sigma^3} - \frac{Q^2(5r^2 + p^2)}{2\Sigma^4}, \\ C_0 &= \frac{4Mrp}{\Sigma^3} + \frac{2Qab(r^2 - p^2)}{rp\Sigma^3} - \frac{Q^2(3r^2 + 2p^2)p}{r\Sigma^4}, \\ C_1 &= \frac{2Qrp}{\Sigma^3} \sqrt{\frac{\Delta_r}{\Sigma}}, \quad C_2 = -\frac{2Qr^2}{\Sigma^3} \sqrt{\frac{\Delta_p}{\Sigma}}, \quad C_3 = -\frac{2Qrp}{\Sigma^3} \sqrt{\frac{\Delta_p}{\Sigma}}, \quad C_4 = \frac{2Qp^2}{\Sigma^3} \sqrt{\frac{\Delta_r}{\Sigma}}. \end{aligned}$$

The non-vanishing components of the Weyl tensor are given by the following one real, two complex scalars

$$\frac{1}{4}(A - D) = -\frac{C}{2} = \frac{M}{\Sigma^2} - \frac{5Q^2}{4\Sigma^3}, \quad (\text{A14})$$

$$\begin{aligned} \Psi_2 &= \frac{1}{4}(A + D) + iC_0 = -B + iC_0 \\ &= \frac{2M}{(r - ip)^2 \Sigma} - \frac{Q^2(5r - 4ip)}{2r(r - ip)^2 \Sigma^2} + \frac{2iQab}{rp(r - ip)^2 \Sigma}, \end{aligned} \quad (\text{A15})$$

$$\frac{C_1 + iC_4}{\sqrt{2}} = \frac{Qp\sqrt{2\Delta_r}}{(r - ip)\Sigma^{5/2}}, \quad (\text{A16})$$

$$\frac{C_2 + iC_3}{\sqrt{2}} = \frac{-Qr\sqrt{2\Delta_p}}{(r - ip)\Sigma^{5/2}}, \quad (\text{A17})$$

while the only non-zero Maxwell scalar is

$$F_{01} + iF_{23} = \frac{\sqrt{3}Q}{(r + ip)\Sigma}. \quad (\text{A18})$$

These scalar invariants characterize the properties of the  $D = 5$  EMCS-Kerr-AdS spacetime.

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